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The Weak Nonlinear Instability of Euler Explicit Scheme for the Convective Equation

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The weak nonlinear instability of the Euler explicit scheme for the convective equation $u_t + uu_x = \nu u_{xx}$ is studied analytically using perturbation expansion and discrete Fourier transform techniques. Short time weak nonlinear growth of the Fourier modes for the noise is found, provided that the linear stability conditions are satisfied. As long as the perturbation expansion is valid, the maximum growth is approximately proportional to the inverse of the grid spacing.

INTRODUCTION

In applying finite difference techniques to the solution of partial differential equations, the stability properties of the resulting difference equations seriously influence the progress of the calculation. As noted by Daly [1], if the finite difference approximation to linear equations with constant coefficients is unstable, fluctuations present in the system continuously grow in amplitude. When the finite difference approximation to nonlinear equations is not stable, the rate of growth of fluctuations is subject to change. The nonlinear terms may speed up the growth rate or establish an upper limit for these fluctuations. These nonlinear phenomena are similar to hydraulic nonlinear stability.

Linear stability has been studied extensively [2, 3]. However, due to mathematical difficulty, these are relatively few published reports on nonlinear stability theory [1, 4, 5, 6]. For nonlinear equations, transition from linear instability to nonlinear instability produces weak nonlinear instability. Though very narrow, this region exhibits nonlinear error growth, and gives an estimation of the upper limit of linear stability. Many problems in this region can be solved analytically by perturbation expansion.

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In general, convective equations are nonlinear and their stability behavior is difficult to describe analytically. A reasonable approach assumes that the time and spatial variations of the field velocity are small compared to the noise variation. Then the field velocity is held constant so that the problem becomes one with constant coefficients for linear stability analysis. This approach is simple and generates much useful information.

In this work, the discrete Fourier transformation and perturbation expansion in a discrete domain are employed to learn about the weak nonlinear instability of the Euler explicit method for $u_t + uu_x = \nu u_{xx}$. Extension of the present method to multiple dimensional problems is straightforward.

ANALYSIS

Consider Burgers' equation

$$u_t + uu_x = \nu u_{xx} \tag{1}$$

where $\nu > 0$. The field velocity is subject to periodic boundary and initial conditions

$$u(t, x) = u(t, x+1) \tag{2}$$

$$u(0, x) = f(x) \tag{3}$$

where $f(x)$ is spatially periodic. Writing the convective term in conservative form, the Euler explicit scheme changes Eq. (1) to

$$\begin{gathered} u(m+1, n) - u(m, n) + \left(\frac{\sigma}{4}\right)\{u^2(m, n+1) - u^2(m, n-1)\} \\ -\theta\{u(m, n+1) - 2u(m, n) + u(m, n-1)\} = 0 \\ 0 \leqslant n < N \\ m = 0, 1, 2, \ldots, \\ N\Delta x = 1 \end{gathered} \tag{4}$$

where $\sigma = \Delta t/\Delta x$, $\theta = \nu\Delta t/\Delta x$, $u(m, n) = u(m\Delta t, n\Delta x)$, and Δt is the integration time interval. The boundary condition becomes

$$u(m, -1) = u(m, N-1). \tag{5}$$

Suppose that a finite difference solution U for field velocity without error or noise, which of course satisfies Eq. (4) and Eq. (5), is known. A small error $v(0, n)$, that satisfies Eq. (5), is added to the initial condition to investigate how it influences the finite difference solution. The finite difference solution becomes

$$u(m, n) = U + v(m, n) \tag{6}$$

where $v(m, n)$ is the result of introducing an error. For the sake of simplicity, the time and spatial variations of U are assumed to be much smaller than the error and to be constant. Substituting Eq. (6) into Eq. (4) and Eq. (5) generates

$$\begin{aligned} v(m+1, n) - v(m, n) + \frac{\alpha}{2} \{v(m, n+1) - v(m, n-1)\} \\ + \frac{\sigma}{4} \{v^2(m, n+1) - v^2(m, n-1)\} \\ - \theta\{v(m, n+1) - 2v(m, n) + v(m, n-1)\} = 0 \end{aligned} \tag{7}$$

$$v(m, 0) = v(m, N) \tag{8}$$

where $\alpha = U\Delta t/\Delta x$. Now the following discrete Fourier transforms are employed.

$$v(m, n) = \sum_{p=0}^{N-1} A(m, p)\, e^{i2np\pi/N}. \tag{9}$$

Then Eq. (8) is satisfied automatically. Using Eq. (9) and the following formula

$$\begin{aligned} &\left\{\sum_{p=0}^{N-1} a(p)\, e^{i2np\pi/N}\right\}\left\{\sum_{q=0}^{N-1} b(q)\, e^{i2nq\pi/N}\right\} \\ &= \sum_{p=0}^{N-1} \left\{\sum_{s=0}^{p} a(s)\, b(p-s) + \sum_{s=p+1}^{N-1} a(s)\, b(N+p-s)\right\} e^{i2np\pi/N} \end{aligned} \tag{10}$$

we transform Eq. (7) into the following amplitude equations:

$$\begin{aligned} &A(m+1, p) - \left\{1 - 2\theta\left(1 - \cos\frac{2p\pi}{N}\right) - i\alpha \sin\frac{2p\pi}{N}\right\} A(m, p) \\ &+ \frac{i\sigma}{2} \sin\frac{2p\pi}{N} \left\{\sum_{s=0}^{p} A(m, s)\, A(m, p-s) + \sum_{s=p+1}^{N-1} A(m, s)\, A(m, N+p-s)\right\} = 0, \\ &\qquad p = 0, 1, 2, \ldots, N-1. \end{aligned} \tag{11}$$

Several interesting facts related to Eq. (10) are shown in the Appendix. It is easy to prove that these equations satisfy the following relations:

$$A(m, p) = A(m, N-p)^* \tag{12}$$

where the star denotes the complex conjugate. Equation 11 is a set of coupled nonlinear difference equations and is very difficult to solve analytically. When the introduced error $v(0, n)$ is small, say smaller than U by one order of magnitude or more, there is a period during which the error $v(m, n)$ remains small. The period determines the range of the weak nonlinear instability. Though this range may be narrow, it controls whether the error will further grow to the range of nonlinear instability or not. Assume that the initial error $v(0, n)$ is small enough so that the following perturbation expansion of $A(m, p)$ is valid.

$$\begin{aligned} A(m, p) &= \varepsilon A_1(m, p) + \varepsilon^2 A_2(m, p) + \cdots \\ A_i(0, p) &= 0, \qquad i > 1, \end{aligned} \tag{13}$$

where ε is a sufficiently small parameter and $A_i(0, p)$'s are of order unity. Substituting Eq. (13) in the amplitude equations produces the following set of decoupled linear difference equations:

$$A_1(m+1, p) - \left\{1 - 2\theta\left(1 - \cos\frac{2p\pi}{N}\right) - i\alpha \sin\frac{2p\pi}{N}\right\} A_1(m, p) = 0$$

$$\begin{aligned} A_2(m+1, p) &- \left\{1 - 2\theta\left(1 - \cos\frac{2p\pi}{N}\right) - i\alpha \sin\frac{2p\pi}{N}\right\} A_2(m, p) \\ &+ \frac{i\sigma}{2}\sin\frac{2p\pi}{N}\left\{\sum_{s=0}^{p} A_1(m, s)\, A_1(m, p-s)\right. \\ &\left. + \sum_{s=p+1}^{N-1} A_1(m, s)\, A_1(m, N+p-s)\right\} = 0. \end{aligned} \tag{14}$$

The solution of the first equation gives the linear stability condition of the von Neumann method in Ref. [7]:

$$\begin{aligned} A_1(m, p) &= a_1(p)\, e^{-m(k_p + i\phi_p)} \\ e^{-(k_p + i\phi_p)} &= 1 - 2\theta\left(1 - \cos\frac{2p\pi}{N}\right) - i\alpha \sin\frac{2p\pi}{N} \end{aligned} \tag{15}$$

where $a_1(p)$'s are the initial mode amplitudes of $v(0, n)$, and are random variables. The linear stability criteria are

$$2\theta \leqslant 1, \qquad 2v \geqslant U^2 \Delta t. \tag{16}$$

The second-order solution is

$$\begin{aligned} A_2(m, p) = &-\frac{i\sigma}{2}\sin\frac{2p\pi}{N}\left\{[2ma_1(0)\, a_1(p)\, e^{-(k_p + i\phi_p)} - a_2(p)]\, e^{-m(k_p + i\phi_p)}\right. \\ &+ \sum_{s=1}^{p-1} \frac{a_1(s)\, a_1(p-s)}{z_1(s, p)}\, e^{-m[k_s + k_{p-s} + i(\phi_s + \phi_{p-s})]} \\ &\left. + \sum_{s=p+1}^{N-1} \frac{a_1(s)\, a_1(N+p-s)}{z_2(s, p)}\, e^{-m[k_s + k_{N+p-s} + i(\phi_s + \phi_{N+p-s})]}\right\} \\ z_1(s, p) = &\left\{e^{-(k_p + i\phi_p)} - e^{-[k_s + k_{p-s} + i(\phi_s + \phi_{p-s})]}\right\} \\ z_2(s, p) = &\left\{e^{-k_p + i\phi_p)} - e^{-[k_s + k_{N+p-s} + i(\phi_s + \phi_{N+p-s})]}\right\} \\ a_2(p) = &\sum_{s=1}^{p-1} \frac{a_1(s)\, a_1(p-s)}{z_1(s, p)} + \sum_{s=p+1}^{N-1} \frac{a_1(s)\, a_1(N+p-s)}{z_2(s, p)}. \end{aligned} \tag{17}$$

It is easy to see that

$$A_2(m, 0) = A_2(m, N/2) = 0. \tag{18}$$

The third-order solution is

$$\begin{aligned} A_2(m, p) = &-\left(\frac{\sigma}{2}\sin\frac{2p\pi}{N}\right)^2 2a_1(0)\{m^2 a_1(0)\, a_1(p)\, e^{2(k_p + i\phi_p)} \\ &- m[a_2(p)\, e^{(k_p + i\phi_p)} \\ &+ a_1(0)\, a_1(p)\, e^{2(k_p + i\phi_p)}] - \cdots\} + \cdots. \end{aligned} \tag{19}$$

Result and Discussion

When the perturbation $v(m, n)$ is a periodic function with period N, Eqs. (7) and (11) are consistent. In general, except for the linear stability solution, the solutions of Eq. (11) are more complicated than those of Eq. (7). But the Fourier solutions of Eq. (11) suggest much more information in the range of weak nonlinear instability. The amplification factors of the linear solutions are

$$e^{-kp} = \left\{\left[1 - 2\theta\left(1 - \cos\frac{2p\pi}{N}\right)\right]^2 + \left[\alpha\sin\frac{2p\pi}{N}\right]^2\right\}^{1/2}. \tag{20}$$

Equation (17) shows that all the higher order solutions die out as the time level m becomes sufficiently large, provided that the linear stability criteria of Eq. (16) are satisfied and the expansion of Eq. (13) is valid. For a moderate value of m rewrite Eq. (17) as

$$\begin{aligned} A_2(m, p) = &-\frac{i\sigma}{2}\sin\frac{2p\pi}{N}\left\{2ma_1(0)\, a_1(p)\, e^{-(m-1)(k_p + i\phi_p)}\right. \\ &+ \sum_{s=1}^{p-1}\frac{a_1(s)\, a_1(p-s)}{z_1(s, p)}\left[e^{-m[k_s + k_{p-s} + i(\phi_s + \phi_{p-s})]} - e^{-m(k_p + i\phi_p)}\right] \\ &\left.+ \sum_{s=p+1}^{N-1}\frac{a_1(s)\, a_1(N+p-s)}{z_2(s, p)}\left[e^{-m[k_s + k_{N+p-s} + i(\phi_s + \phi_{N+p-s})]} - e^{-m(k_p + i\phi_p)}\right]\right\} \\ = &\, R(m, p) + \sum_{\substack{s=1 \\ s \neq p}}^{N-1} P(m, p, s). \end{aligned} \tag{21}$$

$R(m, p)$ is the resonant solution and $P(m, p, s)$'s are the nonresonant solutions. The resonant phenomenon comes from factor m, and is the result of mode–mode excitation between modes 0 and p. $P(m, p, s)$'s reflect energy transformation between modes through excitation [4]. Note that $a_1(p)$'s are random.

Consider single $R(m, p)$ and $P(m, p, s)$ separately. If the linear amplification factor e^{-k_p} is close to unity, m dominates $R(m, p)$ during the period of small and moderate m, and the corresponding error amplitudes will increase in the period. Consider the largest amplification factor at $p = 1$ and $N - 1$. Note that mode $p = N - 1$ is the complex conjugate of mode $p = 1$. If N is sufficiently large, this factor is very close to unity and is

$$e^{-k_p} \simeq \left\{1 - (2\theta - \alpha^2)\left(\frac{2\pi}{N}\right)^2\right\}^{1/2}. \tag{22}$$

For these two modes, the growth of $|R(m, p)|$ will be significant as shown in Fig. 1. Figures 2 and 3 also show the growing path of several $|R(m, p)|$'s. By simple manipulation, it is easy to show that

$$|R(m, p)|_{\max} \propto \frac{\sigma}{k_p} e^{-1} \sin \frac{2p\pi}{N} \tag{23}$$

$$\tilde{m} = (k_p)^{-1} \tag{24}$$

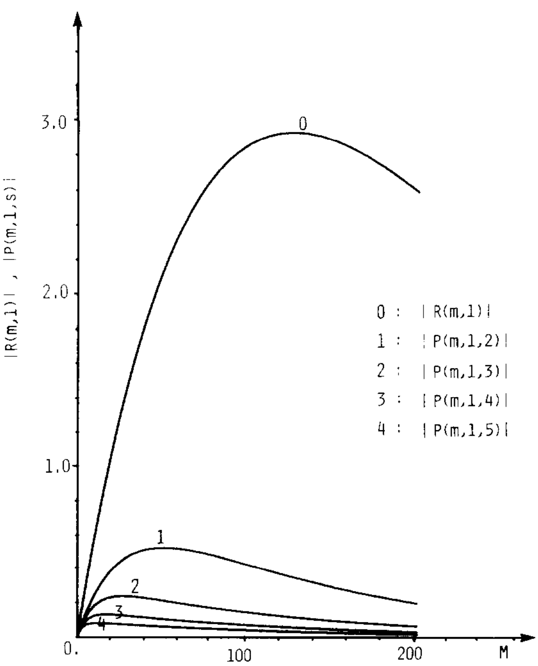

FIG. 1. The growth of resonant and pseudo-resonant of $A_2(m, 1)$, for $N = 40$, $V = a_1(p) = 1$, $\theta = \alpha = 0.4$.

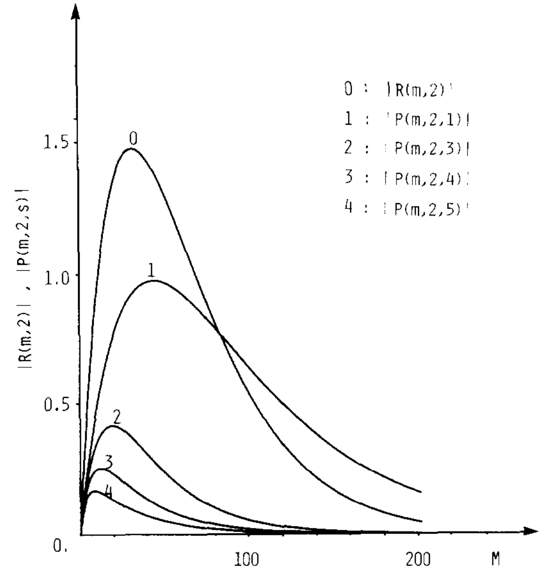

FIG. 2. The growth of resonant and pseudo-resonant of $A_2(m, 2)$, for $N=40$, $V=a_1(p)=1$, $\theta=\alpha=0.4$.

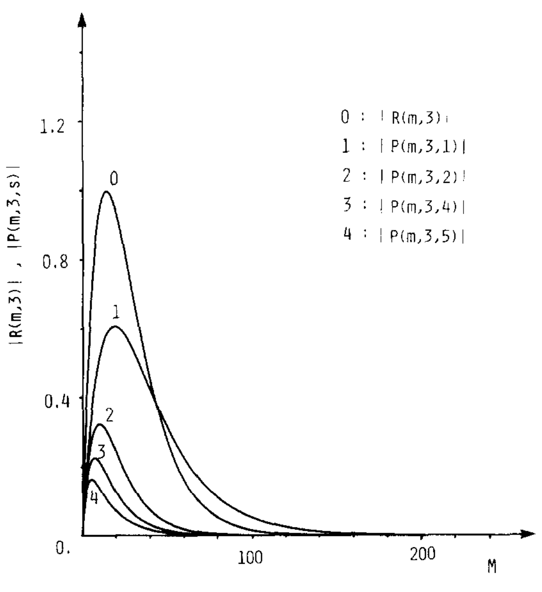

FIG. 3. The growth of resonant and pseudo-resonant of $A_2(m, 3)$, for $N=40$, $V=a_1(p)=1$, $\theta=\alpha=0.4$.

where $\tilde{m}$ is the time step at which $|R(m, p)|$ attains its maximum value and is denoted as a growing period. For small p and sufficiently large N

$$(kp)^{-1} \simeq \frac{N^2}{2(p\pi)^2 (2\theta - a^2)} \tag{25}$$

$$|R(m, p)|_{\max} \propto \frac{N\sigma}{(p\pi)^2 (2\theta - \alpha^2)}. \tag{26}$$

Equation (26) indicates that the second-order solutions corresponding to small p (which are long waves) will grow to an order proportional to N, provided that Eq. (13) is valid. If the magnitude of the initial error modes are $=0.1$ and $N \gg 1$, the strong nonlinear instability region may be switched easily. Table I lists numerical values of the growth peak and growing period for several $R(.)$'s that verify Eqs. (25) and (26). Note that, in addition to the problem of peak and growing period, the number of significant growing modes is also proportional to N.

The growth of the weak nonlinear instability increases as N increases, which corresponds to smaller truncation error. However, if the dispersive or dissipative error is large, which corresponds to large initial error here, the nonlinear stability problem will be switched and Eq. (4) may become nonlinearly unstable. In other words, the following two fundamental criteria of Richtmeyer and Morton [2, pp. 9] may not be satisfied:

1. for fixed Δx, Δt, $\lim_{m \to \infty} |u(m, n) - u(m\,\Delta t, n\,\Delta x)| \to 0$,
2. for fixed $m\,\Delta t$, $\lim_{\Delta t \to 0, \Delta x \to 0} |u(m, n) - u(m\,\Delta t, n\,\Delta x)| \to 0$,

where $u(m\,\Delta t, n\,\Delta x)$ is the exact solution of Eq. (1). The first condition may be violated whenever Δx is small enough, while the possibility of violating the second condition comes from the decrease of Δx. Consequently, it is mathematically impor-

TABLE I

The Growing Peak and Period of $|R(m, p)|$[a]

N	p	$\tilde{m}$[b]	$\lvert R(\tilde{m}, p)\rvert$	N	p	$\tilde{m}$	$\lvert R(\tilde{m}, p)\rvert$
20	1	32	1.4789	100	1	791	7.3216
	2	8	0.7640		2	198	3.6654
	3	3	0.5378		3	88	2.4487
	10	–	–		10	8	0.7640
40	1	126	2.9350	200	1	3167	14.6390
	2	32	1.4789		2	791	7.3216
	3	14	1.0000		3	352	4.8837
	10	–	–		10	32	1.4789

[a] $V = a_1(0) = a_1(p) = 1$, $\theta = \alpha = 0.4$.
[b] Where $\tilde{m}$ is the growing period.

tant to locate the stable region of the nonlinear instability as pointed out by Daly [1] and done by Briggs *et al.* [6].

From Eq. (17), it is seen that the growth is also proportional to $\sigma = \Delta t/\Delta x$. Although the smaller σ has the effect of lowering the unstable character and increasing accuracy in the time domain, it does need longer computing time and increases the risk of accumulating round-off error.

If the initial error distribution makes $a_1(0) = 0$, all $R(m, p)$'s vanish throughout. However, the weak nonlinear instability properties discussed above still exist. There are some $P(m, p, s)$'s with small $z_1(s, p)$ or $z_2(s, p)$ that have as short a time growth as the $R(m, p)$'s have. These $P(m, p, s)$'s are the pseudo-resonant solutions, since either $z_1(s, p) = 0$ or $z_2(s, p) = 0$ corresponds to the resonant solution. Figure 1 demonstrates the growth of $R(m, 1)$ and several $P(m, 1, s)$'s for $N = 40$, $\theta = 0.4$, $\alpha = 0.4$ and $U = 1$. Though the $P(m, 1, s)$'s do not grow so significantly large as $R(m, 1)$, the short time growth is obvious. Figure 2 is for $p = 2$ with similar conditions of Fig. 1. Here the growth of $P(m, 2, 1)$ is the same order as that of $R(m, 2)$. Similar phenomenon can be found in Fig. 3, where $p = 3$. From these figures and Eq. (21), it is clear that the peaks and growing periods of these pseudo-resonances depend on the linear amplification factor e^{-k_p}. Similarly, the number of significant pseudo-resonant solutions is proportional to N. The above discussions indicate that most of the weak nonlinear instability properties are reflected by the resonant behavior. Consequently, the following theorem follows.

THEOREM. *The Euler explicit scheme for Eq. (1) has a short time weak nonlinear error growth behavior, provided that the linear stability criteria are satisfied.*

Another important phenomenon is the initial growing rate. The rate is controlled by e^{-k_p}, σ, and $\sin(2p\pi/N)$. For example, the growing rate of $R(m, 1)$ is not as large as $R(m, 2)$ due to the effect of the sine function. Table II shows several m's at which their corresponding $|R(m, p)|$ is equal to 5, where the factor 5 is considered as one

TABLE II

The Time Step at Which $|R(m, p)| = 5^a$

N	p	m	$\|R(m, p)\|$	N	p	m	$\|R(m, p)\|$
40	1	11	5.3925	200	1	11	5.4957
	2	11	5.0826		2	11	5.4827
	3	13	5.2530		3	11	5.4610
	4	17	5.1340		4	11	5.4309
100	1	11	5.4827				
	2	11	5.4396				
	3	11	5.3459				
	4	11	5.2290				

^{a} $a_1(0) = a_1(p) = 1$, $\theta = 0.4$, $\alpha = 0.8$, $\sigma = 1$.

order of magnitude. The smallest m among all these m's for definite N has physical meaning. If Brandt's multiple level grid scheme [8] is employed, the variation of this m with respect to θ, α, and σ, etc. is a reference for the estimation of the maximum allowable iteration steps for a grid level at each cycle. For fine grid system, the specific m is equal to 11 for the parameters of Table II.

In conclusion, the weak nonlinear growth is linked with the linear growth factor e^{-k_p} and is the result of mode–mode excitation mechanism. Every method that makes e^{-k_p} smaller suppresses the weak nonlinear error growth and eventually reduces the problem of strong nonlinear instability. The condition of using smaller e^{-k_p} can be achieved by adding damping coefficients or properly changing θ, α, and σ. The most successful method for eliminating nonlinear error growth is the multiple level grid scheme with the iteration steps of every grid level being carefully selected. Another powerful and well known method is the scheme using a variable time step at different spatial locations [9]. This method is extremely useful for finding a seady state solution. Other possible methods can be found in [6]. Finally, the present method can be extended to study the iterative solution of the steady state equation provided that the iteration step is interpreted as a time step.

APPENDIX

From Eq. (10), the following facts can be proved easily:

$$\sum_{s=1}^{p} a(s)\, b(p-s) + \sum_{s=p+1}^{N-1} a(s)\, b(N+p-s) = \sum_{s=1}^{p} b(s)\, a(p-s) + \sum_{s=p+1}^{N-1} b(s)\, a(N+p-s) \tag{A.1}$$

$$\sum_{s=1}^{p} a(s)\, b(p-s) + \sum_{s=p+1}^{N-1} a(s)\, b(N+p-s) = \sum_{s=1}^{q} [a(s)\, b(q-s)]^* + \sum_{s=q+1}^{N-1} [a(s)\, b(N+q-s)]^*, \qquad p+q=N, \tag{A.2}$$

By applying Eq. (A.2) to the nonlinear terms of Eq. (11), Eq. (12) follows.

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